11.2

(1) Pf:

Since $f \in C([a,b])$, $\exists x_1, x_2 \in [a,b]$ s.t. $f(x_1) = \inf_{[a,b]} f$, $f(x_2) = \sup_{[a,b]} f$.

WLOG, we may assume that $x_1 \le x_2$,

$$g > 0 \Rightarrow \inf_{[a,b]}(f) \int_{[a,b]} g \le \int_{[a,b]} fg \le \sup_{[a,b]}(f) \int_{[a,b]} g$$

$$\Rightarrow \quad f(x_1) \le \frac{\int_{[a,b]} f \cdot g}{\int_{[a,b]} g} \le f(x_2)$$

$$\Rightarrow \exists c \in [x_1, x_2] \subset [a,b] \text{ s.t. } f(c) = \frac{\int_{[a,b]} fg}{\int_{[a,b]} g}.$$

□

(2) Pf: By (1), $\exists c \in [x_1, x_2] \subset [a,b]$ s.t.

$$f(x_1) \le f(c) = \frac{\int_{[a,b]} fg}{\int_{[a,b]} g} \le f(x_2).$$

If $c = x_1$ or $x_2$, then

$$\int_{[a,b]} fg = f(x_1) \int_{[a,b]} g \quad \text{or} \quad \int_{[a,b]} fg = f(x_2) \int_{[a,b]} g.$$

$$\Rightarrow \int_{[a,b]} (f - \inf f) g = 0 \quad \text{or} \quad \int_{[a,b]} (f - \sup f) g = 0$$

$$\overset{g>0}{\Rightarrow} \quad f \equiv \inf \quad \text{or} \quad g = \sup \text{ on } [a,b]$$

$$\Rightarrow \int_{[a,b]} fg = f(x) \int_{[a,b]} g, \quad \forall x \in [a,b]$$

□

If $c \ne x_1, x_2$, then $c \in (x_1, x_2) \subset (a,b)$, the conclusion also holds.

11.4

Pf: $f \in C([a,b], \mathbb{R}_+) \Rightarrow \exists x_0 \in [a,b]$ s.t. $f(x_0) = \sup_{[a,b]} f$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a,b] \cap [x_0 - \delta, x_0 + \delta]$ s.t. $f(x_0) - \varepsilon \le f(x) \le f(x_0)$.

Hence $(b-a)^{\frac{1}{n}} \sup_{[a,b]} f \geq \left(\int_{[a,b]} f^n\right)^{\frac{1}{n}} \geq \delta^{\frac{1}{n}} \left(\sup_{[a,b]} f - \varepsilon\right)$.

Let $n \to \infty$, we get

$$\sup_{[a,b]} f \geq \lim_{n\to\infty} \left(\int_{[a,b]} f^n\right)^{\frac{1}{n}} \geq \sup_{[a,b]} f - \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} \left(\int_{[a,b]} f^n\right)^{\frac{1}{n}} = \sup_{[a,b]} f$. $\square$

11.6

(1) pf: Set $M = \sup_{[0,1]} f$, then $\left|n\int_0^\alpha t^n \cdot f(t)\,dt\right| \leq M \cdot n \int_0^\alpha t^n dt = M \cdot \frac{n}{n+1} \cdot \alpha^{n+1}$

$\to 0$ as $n \to \infty$.

(2) pf: $\forall \varepsilon > 0$, since $f \in C([0,1])$ & $f(1) = 0 \Rightarrow \exists \delta \in (0,1)$ s.t. $|f(x)| < \varepsilon$, $\forall x \in (\delta, 1)$.

Then $\left|n\int_0^1 t^n \cdot f(t)\,dt\right|$

$\leq \left|n\int_0^\delta t^n \cdot f(t)\,dt\right| + \left|n\int_\delta^1 t^n \cdot f(t)\,dt\right|$

$\leq M \cdot n \cdot \int_0^\delta t^n \cdot dt + \varepsilon \cdot n \int_\delta^1 t^n dt$

$\leq M \cdot \frac{n}{n+1} \cdot \delta^{n+1} + \varepsilon \to \varepsilon$ as $n \to \infty$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} n\int_0^1 t^n \cdot f(t)\,dt = 0$.

11.8.

Pf: $\ln u_n = \frac{1}{n} \ln\left((n+1)\cdot(n+2)\cdots(n+n)\right)$

$= \frac{1}{n}\left(\ln\left((1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})\right) + \ln(n^n)\right)$

$= \frac{1}{n}\sum_{k=1}^{n} \ln(1+\frac{k}{n}) + \ln n$

$\Rightarrow \ln\left(\frac{u_n}{n}\right) \sim \int_0^1 \ln(1+x)\,dx$

$\Rightarrow u_n \sim e^{\int_0^1 \ln(1+x)\,dx} \cdot n = \frac{4n}{e}$

11.10.

(1). $\int_a^b f(t) e^{i\lambda t}\,dt$

$$= \int_a^b f(t)\, d\left(\frac{e^{i\lambda t}}{i\lambda}\right)$$

$$= f(b)\frac{e^{i\lambda b}}{i\lambda} - f(a)\frac{e^{i\lambda a}}{i\lambda} - \frac{1}{i\lambda}\int_a^b f'(t)\, e^{i\lambda t}\, dt$$

$$\Rightarrow \left|\int_a^b f(t) e^{i\lambda t} dt\right|$$

$$\le \frac{|f(b)|+|f(a)|}{\lambda} + \frac{1}{\lambda}\int_a^b |f'(t)|\, dt \to 0 \quad \text{as } \lambda \to +\infty .$$

(2) Set $f = \sum_{k=1}^{m} \lambda_k \mathbb{1}_{I_k}$ for $I_k \subset [a,b]$, $\lambda_k \in \mathbb{R}$, $\forall\, 1 \le k \le m$.

Then $\left|\int_a^b f(t)\, e^{i\lambda t} dt\right|$

$$= \left|\sum_{k=1}^{m} \lambda_k \int_{I_k} d\left(\frac{e^{i\lambda t}}{i\lambda}\right)\right|$$

$$\le \frac{1}{\lambda}\sum_{k=1}^{m} \lambda_k \cdot |I_k| \to 0 \quad \text{as } \lambda \to +\infty$$

(3) pf: $\forall \varepsilon > 0$, $\exists\, g = \sum_{k=1}^{m} \lambda_k \mathbb{1}_{I_k} \in \mathcal{E}([a,b])$ s.t. $|g - f| < \frac{\varepsilon}{b-a}$. Then

$$\left|\int_a^b f(t) e^{i\lambda t} dt\right|$$

$$\le \int_a^b |f(t) - g(t)|\, dt + \left|\int_a^b g(t)\, e^{i\lambda t}\right| dt$$

$$\le \varepsilon + \left|\int_a^b g(t)\, e^{i\lambda t}\right| dt \to \varepsilon \quad \text{as } \lambda \to +\infty$$

Hence $\lim_{\lambda\to\infty} \int_a^b f(t)\, e^{i\lambda t} dt = 0$ for $f \in CM([a,b])$